

Doubly universal Taylor series on simply connected domains

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Abstract

In this article we deal with the existence of doubly universal Taylor series defined on simply connected domains with respect to any center and we generalize the results of G. Costakis and N. Tsirivas for the unit disk.¹

1 Introduction

Let $\Omega \subset \mathbb{C}$ be an open set. We denote by $H(\Omega)$ the space of functions, holomorphic in Ω , endowed with the topology of uniform convergence on compacta. Moreover, for a compact set $K \subset \mathbb{C}$, we denote

$$\mathcal{A}(K) = \{g \in H(K^o) : g \text{ is continuous on } K\}$$

and

$$\mathcal{M} = \{K \subset \mathbb{C} : K \text{ compact set and } K^c \text{ connected set}\}.$$

Let $\zeta_0 \in \Omega$. It is well known that a function $f \in H(\Omega)$, has a Taylor expansion around ζ_0 , which is valid inside the disk of convergence.

If we denote by $\sum_{n=0}^N \frac{f^{(n)}(\zeta_0)}{n!} (z - \zeta_0)^n$, $N = 1, 2, \dots$ the partial sums of the Taylor expansion of f around ζ_0 , then this sequence of polynomials may have very strong approximation properties outside Ω . To be more specific, let us give the definition of universal Taylor series, as given by V.Nestoridis: A function $f \in H(\Omega)$ is said to belong to the collection $U(\Omega, \zeta_0)$ of functions with universal Taylor series expansions around ζ_0 , if the partial sums $\{S_N(f, \zeta_0)(z), N = 1, 2, \dots\}$ are dense in $\mathcal{A}(K)$, for every $K \in \mathcal{M}$ disjoint

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from Ω (the topology of $A(K)$ is induced by the norm $\|g\|_K = \max_{z \in K} |g(z)|$).

V. Nestoridis [5], [6] has shown that $U(\Omega, \zeta_0) \neq \emptyset$, for any simply connected domain Ω and any point $\zeta_0 \in \Omega$. In particular, he showed that the collection $U(\Omega, \zeta_0)$ is a dense, G_δ subset of the space $H(\Omega)$.

Recently, G. Costakis and N. Tsirivas in [3] worked for $\Omega = \mathbb{D}$ (the open unit disk) and introduced a stronger notion of universality with respect to Taylor series. Let us give the definition of their class.

Definition 1.1. *Let (λ_n) be a strictly increasing sequence of positive integers. A function $f \in H(\mathbb{D})$ belongs to the class $U_{CT}(\mathbb{D}, (\lambda_n))$ if the set $\{(S_n(f, 0)(z), S_{\lambda_n}(f, 0)(z)), n = 1, 2, \dots\}$ is dense in $A(K) \times A(K)$, for every $K \in \mathcal{M}$ disjoint from \mathbb{D} . Such a function f will be called doubly universal Taylor series with respect to the sequences (n) , (λ_n) .*

G. Costakis and N. Tsirivas proved that the class $U_{CT}(\mathbb{D}, (\lambda_n))$ is non-empty, if and only if, $\limsup_n \frac{\lambda_n}{n} = +\infty$.

We study a class of functions, with even stronger approximation properties. Namely,

Definition 1.2. *Let (λ_n) be a strictly increasing sequence of positive integers, Ω a simply connected domain and $\zeta_0 \in \Omega$. A function $f \in H(\Omega)$ belongs to the class $U_{CV}(\Omega, (\lambda_n), \zeta_0)$ if $\{(S_n(f, \zeta_0)(z), S_{\lambda_n}(f, \zeta_0)(z)), n = 1, 2, \dots\}$ is dense in $A(K_1) \times A(K_2)$ for every pair of sets $K_1, K_2 \in \mathcal{M}$, disjoint from Ω .*

We improve the method used in [3] in order to obtain the result for different compact sets K_1, K_2 and we also overcome some technical difficulties to generalize it to any simply connected domain Ω . Our main result is again that $U_{CV}(\Omega, (\lambda_n))$ is non-empty, if and only if, $\limsup_n \frac{\lambda_n}{n} = +\infty$.

For analogous results in other type of universalities we refer to [1], [2]. The article of G. Costakis and N. Tsirivas has complete bibliography on the subject, which features the motivation for investigating such questions.

2 The positive result

Let K, L be compact sets and $f : K \rightarrow \mathbb{C}$ be a continuous function defined on K .

Let p be a polynomial. We denote by $\deg p$ the degree of the polynomial i.e. the highest degree of its terms and by $\deg^- p$ the lowest degree of its

(non-zero) terms. Then for any choice of integers $n > m > 1$ we use the notation:

$$d_{n,m}(f, K, L) = \inf\{\max\{\|f - p\|_K, \|p\|_L\} : p \text{ polynomial such that } \deg^- p \geq m \text{ and } \deg p \leq n\}.$$

The following proposition is crucial for our result. It is a slight modification of the corresponding proposition in [3].

Proposition 2.1. *Let $K, L \in \mathcal{M}$ be two disjoint compact sets such that $0 \in L^\circ$. Let, in addition, $\{\tau_n\}_{n \in \mathbb{N}}$ and $\{\sigma_n\}_{n \in \mathbb{N}}$ be two sequences of positive integers such that $\frac{\tau_n}{\sigma_n} \rightarrow +\infty$ and $U \subset \mathbb{C}$ open, $K \subset U$. Then there exists $\theta \in (0, 1)$ such that for every function f holomorphic in U*

$$\limsup_n d_{\tau_n, \sigma_n}(f, K, L)^{\frac{1}{\tau_n}} < \theta.$$

Proof. It is similar to the proof of theorem 2.1 in [3]. □

Theorem 2.1. *Let Ω be a simply connected domain and $\zeta_0 \in \Omega$. Consider $(\lambda_n)_{n \in \mathbb{N}}$ a strictly increasing sequence of positive integers such that $\limsup_{n \in \mathbb{N}} \frac{\lambda_n}{n} = +\infty$. Then the set $U_{CV}(\Omega, (\lambda_n), \zeta_0)$ is G_δ and dense in $H(\Omega)$.*

Proof. Let $(f_j)_{j \in \mathbb{N}}$ be an enumeration of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ and let $(K_m)_{m \in \mathbb{N}}$ be a sequence of sets in \mathcal{M} , disjoint from Ω such that the following holds: every non-empty compact set $K \subset \mathbb{C} \setminus \Omega$, having connected complement, is contained in some K_m (for the existence of such a sequence we refer to [6]).

For any positive integers m_1, m_2, j_1, j_2, s, n , we denote by $E(m_1, m_2, j_1, j_2, s, n)$ the set

$$E(m_1, m_2, j_1, j_2, s, n) =$$

$$\{f \in H(\Omega) : \|S_n(f, \zeta_0) - f_{j_1}\|_{K_{m_1}} < \frac{1}{s} \text{ and } \|S_{\lambda_n}(f, \zeta_0) - f_{j_2}\|_{K_{m_2}} < \frac{1}{s}\}.$$

Using Mergelyan's theorem it is easy to prove that

$$U_{CV}(\Omega, (\lambda_n), \zeta_0) = \bigcap_{m_1, m_2, j_1, j_2, s \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} E(m_1, m_2, j_1, j_2, s, n).$$

(see for example similar proof in [6]).

Moreover, for every $m_1, m_2, j_1, j_2, s, n \in \mathbb{N}$ the set $E(m_1, m_2, j_1, j_2, s, n)$ is open (see for example [6]).

Therefore, in view of Baire's Category theorem, it suffices to prove that for every choice of positive integers m_1, m_2, j_1, j_2, s the set $\bigcup_{n \in \mathbb{N}} E(m_1, m_2, j_1, j_2, s, n)$ is dense in $H(\Omega)$.

Fix $m_1, m_2, j_1, j_2, s \in \mathbb{N}$. Let $\varepsilon > 0$, $L \subset \Omega$ be a compact set and $g \in H(\Omega)$. Without loss of generality, we may assume that $\zeta_0 \in L^\circ$ and that the compact set L has connected complement (note that Ω is simply connected).

We will prove the existence of a function $f \in H(\Omega)$ and an $n \in \mathbb{N}$, such that:

- $\sup_{z \in L} |f(z) - g(z)| < \varepsilon$
- $\sup_{z \in K_{m_1}} |S_n(f, \zeta_0)(z) - f_{j_1}(z)| < \frac{1}{s}$
- $\sup_{z \in K_{m_2}} |S_{\lambda_n}(f, \zeta_0)(z) - f_{j_2}(z)| < \frac{1}{s}$.

First, we apply Runge's theorem to find a polynomial p such that

$$\sup_{z \in L} |p(z) - g(z)| < \frac{\varepsilon}{2} \text{ and } \sup_{z \in K_{m_1}} |p(z) - f_{j_1}(z)| < \frac{1}{2s}.$$

Since $\limsup_{n \in \mathbb{N}} \frac{\lambda_n}{n} = +\infty$, there exists a strictly increasing sequence of positive integers $(\mu_n)_{n \in \mathbb{N}}$, such that $\frac{\lambda_{\mu_n}}{\mu_n} \rightarrow +\infty$. Thus, $\frac{\lambda_{\mu_n}}{\mu_n + 1} \rightarrow +\infty$, as $n \rightarrow +\infty$.

It is easy to see that the sets $K_{m_2} - \zeta_0 = \{z - \zeta_0, \forall z \in K_{m_2}\}$ and $L - \zeta_0 = \{z - \zeta_0, \forall z \in L\}$ are compact, disjoint and they have connected complements. Applying proposition 2.1, we have that:

$$\limsup_n [d_{\lambda_{\mu_n}, \mu_n + 1}(f_{j_2}(z + \zeta_0) - p(z + \zeta_0), K_{m_2} - \zeta_0, L - \zeta_0)]^{\frac{1}{\lambda_{\mu_n}}} < \theta,$$

for some $\theta \in (0, 1)$.

Therefore, there exists $N \in \mathbb{N}$ such that:

$$[d_{\lambda_{\mu_n}, \mu_n + 1}(f_{j_2}(z + \zeta_0) - p(z + \zeta_0), K_{m_2} - \zeta_0, L - \zeta_0)]^{\frac{1}{\lambda_{\mu_n}}} < \theta, \quad n \geq N.$$

Thus, we may choose a sequence of polynomials P_n , $n \geq N$, with $\deg^- P_n \geq \mu_n + 1$ and $\deg P_n \leq \lambda_{\mu_n}$ such that

$$\sup_{z \in K_{m_2} - \zeta_0} |f_{j_2}(z + \zeta_0) - p(z + \zeta_0) - P_n(z)| \leq \theta^{\lambda_{\mu_n}} \text{ and } \sup_{z \in L - \zeta_0} |P_n(z)| \leq \theta^{\lambda_{\mu_n}}.$$

Let us fix $n_0 \in \mathbb{N}$, $n_0 \geq N$ such that $\mu_{n_0} \geq \deg p(z)$ and $\theta^{\lambda_{\mu_{n_0}}} < \min\{\frac{\varepsilon}{2}, \frac{1}{2s}\}$.

We set $f(z) = P_{n_0}(z - \zeta_0) + p(z)$.

Then

$$S_{\mu_{n_0}}(f, \zeta_0)(z) = S_{\mu_{n_0}}(p, \zeta_0)(z) = p(z).$$

Also,

$$S_{\lambda_{\mu_{n_0}}}(f, \zeta_0)(z) = f(z).$$

Thus,

$$\begin{aligned} \sup_{z \in L} |f(z) - g(z)| &= \sup_{z \in L} |P_{n_0}(z - \zeta_0) + p(z) - g(z)| \leq \\ &\leq \sup_{z \in L} |P_{n_0}(z - \zeta_0)| + \sup_{z \in L} |p(z) - g(z)| = \\ &\sup_{z \in L - \zeta_0} |P_{n_0}(z)| + \sup_{z \in L} |p(z) - g(z)| \leq \theta^{\lambda_{\mu_{n_0}}} + \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

$$\sup_{z \in K_{m_1}} |S_{\mu_{n_0}}(f, \zeta_0)(z) - f_{j_1}(z)| = \sup_{z \in K_{m_1}} |p(z) - f_{j_1}(z)| < \frac{1}{2s} < \varepsilon,$$

$$\begin{aligned} \sup_{z \in K_{m_2}} |S_{\lambda_{\mu_{n_0}}}(f, \zeta_0)(z) - f_{j_2}(z)| &= \sup_{z \in K_{m_2}} |P_{n_0}(z - \zeta_0) + p(z) - f_{j_2}(z)| = \\ &= \sup_{z \in K_{m_2} - \zeta_0} |P_{n_0}(z) + p(z + \zeta_0) - f_{j_2}(z + \zeta_0)| \leq \theta^{\lambda_{\mu_{n_0}}} < \varepsilon. \end{aligned}$$

The function f satisfies all the requirements and the result follows. \square

3 Negative Result

In this section we will use the notation $U_{CT}(\Omega, (\lambda_n))$ for the corresponding class of G. Costakis and N. Tsirivas in any simply connected domain Ω . Obviously $U_{CV}(\Omega, (\lambda_n)) \subset U_{CT}(\Omega, (\lambda_n))$. So, in order to prove that the class $U_{CV}(\Omega, (\lambda_n))$ is otherwise empty we will prove the stronger result that the class $U_{CT}(\Omega, (\lambda_n))$ is empty. For this we need the following result of J. Müller and A. Yavrian in [4].

Theorem 3.1. (*Müller–Yavrian*) *Let Γ be a compact and connected subset of \mathbb{C} , but not a singleton. Let $E \subset \mathbb{C}$ be a closed set such that E is non-thin at ∞ . Also suppose (P_n) to be a sequence of polynomials with $\deg P_n \leq d_n$, for some increasing sequence of positive integers (d_n) and having the following properties:*

(α) there exists a function $f : \Gamma \rightarrow \mathbb{C}$ with

$$\limsup_{n \rightarrow +\infty} \|f - P_n\|_{\Gamma}^{1/d_n} < 1,$$

(β) for all $z \in E$

$$\limsup_{n \rightarrow +\infty} |P_n(z)|^{1/d_n} \leq 1.$$

Then the following statement is true:

(i) if the sequence (d_{n+1}/d_n) is bounded, then f extends to an entire function and for every compact set $K \subset \mathbb{C}$ we have

$$\limsup_{n \rightarrow +\infty} \|f - P_n\|_K^{1/d_n} < 1,$$

(ii) if, for arbitrary (d_n) , the function f is analytic on Γ , then f extends to a holomorphic function having a simply connected domain of existence $G_f \subset \mathbb{C}$ (G_f denotes the unique largest domain on which f extends as a holomorphic function; observe that G_f exists in this case) and for every compact set $K \subset G_f$ we have

$$\limsup_{n \rightarrow +\infty} \|f - P_n\|_K^{1/d_n} < 1.$$

Theorem 3.2. Let $\Omega \subset \mathbb{C}$ be a simply connected domain and $\zeta_0 \in \Omega$. If $(\lambda_n)_{n \in \mathbb{N}}$ is a strictly increasing sequence of positive integers such that $\limsup_{n \in \mathbb{N}} \frac{\lambda_n}{n} < +\infty$, then $U_{CT}(\Omega, (\lambda_n), \zeta_0) = \emptyset$.

Proof. Assume first that Ω is not bounded. Arguing by contradiction, suppose that there exists $f \in U_{CT}(\Omega, (\lambda_n), \zeta_0)$. Since the sequence $(\frac{\lambda_n}{n})_{n \in \mathbb{N}}$ is bounded, there exists $C > 0$ such that $\frac{\lambda_n}{n} < C$, for every $n \in \mathbb{N}$. We consider the sets

$$E_n = \Omega^c \cap (D(\zeta_0, 2^C))^c \cap \overline{D(\zeta_0, 2^C + n)}, \quad n = 1, 2, \dots$$

(we denote by $D(z, r)$ the open disk of center z and radius r).

Without loss of generality, we may assume that $E_n \neq \emptyset$, $n \geq 1$, since this is eventually the case anyway (or by choosing suitable $C > 0$).

For every $n \in \mathbb{N}$, it is easy to see that the set $E_n \subset \Omega^c$ is closed and bounded, thus is compact. Moreover, each E_n has connected complement. To see this

note that $E_n^c = \Omega \cup D(\zeta_0, 2^C) \cup \overline{D(\zeta_0, 2^C + n)}^c$, $\zeta_0 \in \Omega \cap D(\zeta_0, 2^C)$ and Ω is unbounded.

Let $E = \bigcup_{n \in \mathbb{N}} E_n = \Omega^c \cap D(\zeta_0, 2^C)^c$. Since Ω^c is connected, it is non-thin at ∞ (see page 79 theorem 3.8.3 in [7]). Because thinness is a local property (see definition page 79 in [7]), E is also non-thin at ∞ .

Now we use the fact that f belongs to the class $U_{CT}(\Omega, (\lambda_n), \zeta_0)$, to fix an $n_1 \in \mathbb{N}$, such that

$$\sup_{z \in E_1 \cup \{1+\zeta_0\}} |S_{n_1}(f, \zeta_0)(z)| < \frac{1}{2^{C+1}} \text{ and } \sup_{z \in E_1 \cup \{1+\zeta_0\}} |S_{\lambda_{n_1}}(f, \zeta_0)(z) - 1| < \frac{1}{2^{C+1}}.$$

Thus,

$$\begin{aligned} & \sup_{z \in E_1 \cup \{1+\zeta_0\}} |S_{\lambda_{n_1}}(f, \zeta_0)(z) - S_{n_1}(f, \zeta_0)(z) - 1| \leq \\ & \leq \sup_{z \in E_1 \cup \{1+\zeta_0\}} |S_{n_1}(f, \zeta_0)(z)| + \sup_{z \in E_1 \cup \{1+\zeta_0\}} |S_{\lambda_{n_1}}(f, \zeta_0)(z) - 1| < \frac{1}{2^C}. \end{aligned}$$

For every $z \in \mathbb{C}$, we set

$$S_{\lambda_{n_1}}(f, \zeta_0)(z) - S_{n_1}(f, \zeta_0)(z) = (z - \zeta_0)^{n_1+1} P_1(z),$$

where P_1 is a polynomial with

$$\deg P_1 \leq \lambda_{n_1} - (n_1 + 1) < \lambda_{n_1} - n_1 = n_1 \left(\frac{\lambda_{n_1}}{n_1} - 1 \right) \leq n_1(C - 1).$$

Consequently,

$$\sup_{z \in E_1 \cup \{1+\zeta_0\}} |(z - \zeta_0)^{n_1+1} P_1(z) - 1| < \frac{1}{2^C}.$$

Also, since $z \in E_1$ implies that $|z - \zeta_0| > 2^c$ we have:

$$\begin{aligned} & \sup_{z \in E_1} |P_1(z)| \leq \sup_{z \in E_1} \frac{1}{|(z - \zeta_0)^{n_1+1}|} \sup_{z \in E_1} |(z - \zeta_0)^{n_1+1} P_1(z)| \leq \\ & \leq \frac{1}{2^{C(n_1+1)}} \left(\sup_{z \in E_1} |(z - \zeta_0)^{n_1+1} P_1(z) - 1| + 1 \right) \leq \frac{1}{2^{C(n_1+1)}} \left(\frac{1}{2^C} + 1 \right) = \\ & = \frac{1}{2^C n_1} \left(\frac{1}{4^C} + \frac{1}{2^C} \right) < \frac{1}{2^{C n_1}}. \end{aligned}$$

Repeating the above argument, we fix an $n_2 \in \mathbb{N}$, with $n_2 > n_1$, such that

$$\sup_{z \in E_2 \cup \{1+\zeta_0\}} |S_{n_2}(f, \zeta_0)(z)| < \frac{1}{2^{C+2}} \text{ and } \sup_{z \in E_2 \cup \{1+\zeta_0\}} |S_{\lambda_{n_2}}(f, \zeta_0)(z) - 1| < \frac{1}{2^{C+2}}.$$

As before, we set

$$\frac{S_{\lambda_{n_2}}(f, \zeta_0)(z) - S_{n_2}(f, \zeta_0)(z)}{(z - \zeta_0)^{n_2+1}} = P_2(z).$$

Then P_2 is a polynomial of degree less than $n_2(C - 1)$.

Moreover as before,

$$\sup_{z \in E_2 \cup \{1+\zeta_0\}} |(z - \zeta_0)^{n_2+1} P_2(z) - 1| < \frac{1}{2^{C+1}}.$$

And,

$$\sup_{z \in E_2} |P_2(z)| < \frac{1}{2^{C n_2}}.$$

Proceeding inductively, we conclude that for every $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$, with $n_k > n_{k-1}$ and polynomial P_k , $\deg P_k \leq C n_k$ such that

$$\sup_{z \in E_k \cup \{1+\zeta_0\}} |(z - \zeta_0)^{n_k+1} P_k(z) - 1| < \frac{1}{2^{C+k-1}} \quad (1)$$

and

$$\sup_{z \in E_k} |P_k(z)| \leq \frac{1}{2^{C n_k}}.$$

Let $z \in E$. Then

$$\limsup_{k \rightarrow \infty} |P_k(z)|^{1/C n_k} \leq \lim_{k \rightarrow \infty} \frac{1}{2} = \frac{1}{2}. \quad (2)$$

(Note that, $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of compact sets.)

Also, since $\sup_{z \in E_1} |P_k(z)| \leq \sup_{z \in E_k} |P_k(z)| \leq \frac{1}{2^{C n_k}}$, for every $k \in \mathbb{N}$

$$\limsup_{k \rightarrow \infty} \|P_k(z)\|_{E_1}^{1/C n_k} \leq \lim_{k \rightarrow \infty} \left(\frac{1}{2^{C n_k}} \right)^{1/C n_k} = \frac{1}{2} < 1. \quad (3)$$

Therefore, we may apply the theorem *Müller – Yavrian*, to a compact and connected subset Γ of E_1 (containing more than one points), the closed set E , which is non-thin at ∞ , the sequence of polynomials P_k , $d_k = C n_k$, $k \in \mathbb{N}$, and the function $f \equiv 0$. From inequalities (2), (3), we observe that the conditions of the theorem are fulfilled. Since the function $f \equiv 0$ is entire, it follows that $P_k \rightarrow 0$, uniformly on all compact subsets of \mathbb{C} . In particular, $P_k(1 + \zeta_0) \rightarrow 0$, as $k \rightarrow \infty$. On the other hand, (1) implies that

$P_k(1 + \zeta_0) \rightarrow 1$, as $k \rightarrow \infty$, which is a contradiction. As a result, in case that Ω is not bounded, the class $U_{CT}(\Omega, (\lambda_n), \zeta_0)$ is empty.

Now if the set Ω is bounded, then $\Omega \cup D(\zeta_0, 2^C) \subset D(0, N)$, for some $N \in \mathbb{N}$. One can apply the previous procedure for the compact sets $E_n = [N + 1, N + n]$, for $n > 1$ and the result follows. \square

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